Digital Communication Systems
ECS 452

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7. The Waveform Channel

Office Hours:

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Monday	10:00-10:40
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The Big Plan

- Although we are thinking about digital communication systems,
- actual signaling in the wire or air is in continuous time which is described by the **waveform channel**:

$$R(t) = S(t) + N(t).$$

- Directly finding the optimal (MAP) detector or evaluating the performance $P(\mathcal{E})$ of such system is difficult.
- Our approach is to first construct an equivalent **vector channel** that preserves the relevant features. (Chapter 7)
- Then, at the end, we can use what we learn to go back to the original waveform channel and directly work with the waveforms.

Review: ECS315

ECS 315: Probability and Random Processes **2017/1**

HW Solution 12 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

Problem 4 (Randomly Phased Sinusoid)**.** Suppose Θ is a uniform random variable on the interval $(0, 2\pi)$.

(a) Consider another random variable X defined by

$$X = 5\cos(7t + \Theta)$$

where t is some constant. Find $\mathbb{E}[X]$.

Solution: First, because Θ is a uniform random variable on the interval $(0, 2\pi)$, we know that $f_\Theta(\theta) = \frac{1}{2\pi}1_{(0,2\pi)}(t)$. Therefore, for "any" function g, we have

$$\mathbb{E}[g(\Theta)] = \int_{-\infty}^{\infty} g(\theta) f_\Theta(\theta) d\theta.$$

(a) X is a function of Θ. $\mathbb{E}[X] = 5\mathbb{E}[\cos(7t + \Theta)] = 5\int_0^{2\pi} \frac{1}{2\pi}\cos(7t + \theta)d\theta$. Now, we know that integration over a cycle of a sinusoid gives 0. So, $\mathbb{E}[X] = \boxed{0}$.

[ECS315 2017]

Review: ECS315

11.4 Linear Dependence

Definition 11.47. Given two random variables X and Y, we may calculate the following quantities:

(a) **Correlation**: $\mathbb{E}[XY]$. $= \sum_x \sum_y xy\, p_{X,Y}(x,y) = \mathbb{E}[YX]$

(b) **Covariance**: $\text{Cov}[X,Y] = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$.

(c) **Correlation coefficient**: $\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y}$

Exercise 11.48 (F2011). Continue from Exercise 11.7.

(a) Find $\mathbb{E}[XY]$.

(b) Check that $\text{Cov}[X,Y] = -\frac{1}{25}$.

$\text{Cov}[X,X] = \mathbb{E}[XX] - \mathbb{E}X\,\mathbb{E}X$

$= \mathbb{E}[X^2] - (\mathbb{E}X)^2$

11.49. $\text{Cov}[X,Y] = \mathbb{E}[(X - \underbrace{\mathbb{E}X}_{m_X})(Y - \underbrace{\mathbb{E}Y}_{m_Y})] = \mathbb{E}[XY] - \mathbb{E}X\mathbb{E}Y$

$= \mathbb{E}[(X - m_X)(Y - m_Y)] = \mathbb{E}[XY - Ym_X - Xm_Y + m_X m_Y]$

$= \mathbb{E}[XY] - m_X\mathbb{E}Y - m_Y\mathbb{E}X + m_X m_Y$

$\text{Cov}[X,X]$

$= \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)]$

$= \mathbb{E}[(X - \mathbb{E}X)^2]$

$= \text{Var}\, X$

- Note that $\text{Var}\, X = \text{Cov}[X,X]$.

11.50. $\text{Var}[X+Y] = \text{Var}\, X + \text{Var}\, Y + 2\text{Cov}[X,Y]$

[ECS315 2017]

Review: ECS315

ECS 315: Probability and Random Processes 2017/1

HW Solution 12 — Due: Not Due

Lecturer: Prapun Suksompong, Ph.D.

(b) Consider another random variable Y defined by

$$Y = 5\cos(7t_1 + \Theta) \times 5\cos(7t_2 + \Theta)$$

where t_1 and t_2 are some constants. Find $\mathbb{E}[Y]$.

Solution:

(b) Y is another function of Θ.

$$\begin{aligned}\mathbb{E}[Y] &= \mathbb{E}[5\cos(7t_1+\Theta)\times 5\cos(7t_2+\Theta)] = \int_0^{2\pi} \frac{1}{2\pi} 5\cos(7t_1+\theta)\times 5\cos(7t_2+\theta)d\theta \\ &= \frac{25}{2\pi}\int_0^{2\pi}\cos(7t_1+\theta)\times\cos(7t_2+\theta)d\theta.\end{aligned}$$

Recall[1] the cosine identity

$$\cos(a)\times\cos(b) = \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right).$$

Therefore,

$$\begin{aligned}\mathbb{E}Y &= \frac{25}{4\pi}\int_0^{2\pi}\cos(14t+2\theta)+\cos(7(t_1-t_2))\,d\theta \\ &= \frac{25}{4\pi}\left(\int_0^{2\pi}\cos(14t+2\theta)\,d\theta + \int_0^{2\pi}\cos(7(t_1-t_2))\,d\theta\right).\end{aligned}$$

The first integral gives 0 because it is an integration over two period of a sinusoid. The integrand in the second integral is a constant. So,

$$\mathbb{E}Y = \frac{25}{4\pi}\cos(7(t_1-t_2))\int_0^{2\pi}d\theta = \frac{25}{4\pi}\cos(7(t_1-t_2))\,2\pi = \boxed{\frac{25}{2}\cos(7(t_1-t_2))}.$$

[ECS315 2017]